Hilbert Spaces

Ryan T Whyte

Fullerton College

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Outline

Question

- Are there natural, separable Hilbert Spaces on the Euclidean Ball for which all composition operators are bounded?

Background

- David Hilbert (1862-1943)

Binary Operators

- Definition: "*" is called a *binary operation* on a set A iff $* : A \times A \to A$.

- Example: $- : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $-(x, y) := x - y$
  Since $- : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, where $\mathbb{R} = \mathbb{R}$, "-" is a binary operator on $\mathbb{R}$.

Composition Operators

- Definition: "C_ϕ" is called a *composition operation* on a function f iff $C_\phi(f) = f \circ \phi$ where $f \circ \phi$ denotes usual function composition.
- Example: Let $f : Y \to Z$ and $g : X \to Y$.
  Then $C_g(f) = f \circ g : X \to Z$

Field

- Definition: We call $\mathcal{F}$ a *field* iff $\mathcal{F} = (F, +, \cdot)$, where:
  - F is a non-empty set;
  - $+$ is a binary operation on F;
  - $\forall a, b, c \in F, (a + b) + c = a + (b + c)$;
  - $\forall a, b, c \in F, (a \cdot b) \cdot c = a \cdot (b \cdot c)$;
  - $\forall a, b \in F, a + b = b + a$;
  - $\forall a, b \in F, ab = ba$;
  - $\forall a, b, c \in F, a(b + c) = ab + ac$;
  - $\forall a, b, c \in F, (a + b)c = ac + bc$;
  - $\exists 0_F \in F$ such that $x + 0_F = x = 0_F + x, \forall x \in F$;
  - $\exists 1_F \in F$ such that $1_F \neq 0_F$ and $a \cdot 1_F = a = 1_F \cdot a$, where $a \in \mathbb{C}$;
  - $\forall x \in F$ such that $x \neq 0_F$, $\exists x^{-1} \in F$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1_F$;
  - $\forall x \in F$, $\exists -x \in F$ such that $x + (-x) = 0_F = (-x) + x$.

Field Example

- Suppose that $\mathscr{J} = (\mathbb{C}, +, \cdot)$, where $+$ and $\cdot$ are usual complex addition and complex multiplication, and $\mathbb{C}$ is the set of all complex numbers. Show that $(\mathbb{C}, +, \cdot)$ is a field.
  - Let $a, b, c \in \mathbb{C}$
  - $2 + 7\imath \in \mathbb{C} \therefore \mathbb{C}$ is a non-empty set.
  - By the Closure Property of usual complex number addition, $+$ is a binary operator on $\mathbb{C}$, and by Closure Property of usual complex number multiplication, $\cdot$ is a binary operator on $\mathbb{C}$
  - From Algebra 2 we know that the following hold true:
    - $(a + b) + c = a + (b + c)$;
    - $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;
    - $a + b = b + a$;
    - $ab = ba$;
    - $a(b + c) = ab + ac$;
    - $(a + b)c = ac + bc$;
    - $a + (-a) = 0 + 0\imath = 0$
    - $a \cdot \left(\frac{1}{a}\right) = 1 + 0\imath = 1$
  - Let $0_{\mathbb{C}} := 0 + 0\imath$, then $a + 0 + 0\imath = a$. This implies $a + 0_{\mathbb{C}} = a$;
  - Let $1_{\mathbb{C}} := 1 + 0\imath$, then $a \cdot (1 + 0\imath) = a + a \cdot 0\imath = a$. Note that $1 + 0\imath \neq 0 + 0\imath$, $\therefore 1_{\mathbb{C}} \neq 0_{\mathbb{C}}$. This implies $a \cdot 1_{\mathbb{C}} = a$;

Vector Spaces

- Definition: Let $\mathcal{F} = (F, +, \cdot)$ be a field. We call $\mathcal{V}$ an an $\mathcal{F}$-vector space, or, alternatively, a vector space over $\mathcal{F}$, iff $\mathcal{V} = (V, \mathcal{F}, \oplus, \bullet)$ and the following axioms hold:
  - V is a non-empty set. (Non-emptiness Property)
  - $\oplus$ is a binary operation on V. (Closure Property of Addition)
  - $\bullet$ is an F-multiplication on V. (Closure Property of $\bullet$)
  - $\exists 0_V \in V$ such that $v \oplus 0_V = 0_V \oplus v = v \; \forall v \in V$. (Additive Identity Property)
  - $\forall v \in V \; \exists(-v) \in V$ such that $v \oplus (-v) = (-v) \oplus v = 0_V$. (Additive Inverse Property)
  - $v \oplus w = w \oplus v \; \forall v, w \in V$. (Commutative Property of $\oplus$)
  - $u \oplus (v \oplus w) = (u \oplus v) \oplus w \; \forall u, v, w \in V$. (Associative Property of $\oplus$)
  - $1_F v = v \; \forall v \in V$. (Multiplicative Identity Property)
  - If $\alpha, \beta \in F$ and $v \in V$, then $(\alpha + \beta)v = \alpha v \oplus \beta v$. (Distributive Property of $\mathcal{F}$ over V)
  - $\forall \alpha \in F$, $u, v \in V$, we have that $\alpha(u \oplus v) = \alpha u \oplus \alpha v$. (Distributive Property of $\mathcal{V}$ over $\mathcal{F}$)
  - $\forall \alpha, \beta \in F$ and $v \in V$, we have that $(\alpha\beta)v = \alpha(\beta v)$. (Associative Property of F-multiplication on V)
- We call a vector space $\mathcal{V}$ *real* iff $\mathcal{V} = (V, (\mathbb{R}, +, \cdot), \oplus, \bullet)$, where $+$ and $\cdot$ denote usual real addition and multiplication, respectively.
- We call a vector space $\mathcal{V}$ *complex* iff $\mathcal{V} = (V, (\mathbb{C}, +, \cdot), \oplus, \bullet)$, where $+$ and $\cdot$ denote usual complex addition and multiplication, respectively.

Vector Space Example

- Let $\mathcal{E} := (\mathbb{C}, \mathcal{J}, +, \cdot)$, where $+$ and $\cdot$ are usual complex addition and complex multiplication, respectively, and $\mathcal{J}(\mathbb{C}, +, \cdot)$ is the previously mentioned field. Let's prove that $\mathcal{V}$ is an $\mathcal{J}$-vector space.
  - Let $\alpha, \beta \in \mathcal{J}$ and $u, v, w \in \mathbb{C}$
  - $2 + 7\imath \in \mathbb{C} \therefore \mathbb{C}$ is a non-empty set.
  - By the Closure Property of usual complex number addition, $+$ is a binary operator on $\mathbb{C}$, and by the Closure Property of usual complex number multiplication, $\cdot$ is a binary operator on $\mathbb{C}$
  - From Algebra 2 we know that the following hold true:
    - $v + (-v) = 0 + 0\imath = 0$;
    - $v + w = w + v$;
    - $u + (v + w) = (u + v) + w$;
    - $\alpha \cdot (v + w) = \alpha v + \alpha w$;
    - $(\alpha + \beta) \cdot v = \alpha v + \beta v$;
    - $(\alpha \cdot \beta) \cdot v = \alpha v \cdot \beta v$
  - Let $0_{\mathbb{C}} := 0 + 0\imath$, then $v + 0 + 0\imath = v$. This implies $v + 0_{\mathbb{C}} = v$;
  - Let $1_{\mathbb{C}} := 1 + 0\imath$, then $v \cdot (1 + 0\imath) = v + v \cdot 0\imath = v$. Note that $1 + 0\imath \neq 0 + 0\imath$, $\therefore 1_{\mathbb{C}} \neq 0_{\mathbb{C}}$. This implies $v \cdot 1_{\mathbb{C}} = v$;

Inner Product Spaces

- Definition: Let $\mathcal{E}$ be a complex vector space. A mapping $\langle \cdot, \cdot \rangle E \times E \to \mathbb{C}$ is called an inner product space in E if $\forall x, y, z \in E$ and $a \in \mathbb{C}$ the following conditions are satisfied:
  - $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (Hermetian Symmetric);
  - $\langle a \cdot x, y \rangle = \alpha \cdot \langle x, y \rangle$ (Linearity in the first argument);
  - $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$;
  - $\langle x, x \rangle \geq 0$ (Positive-definiteness);
  - $\langle x, x \rangle = 0$ iff $x = 0$;

Inner Product Example

- Prove the following statements:
- Let $\mathcal{C}^n = (\mathbb{C}^n, \oplus, \odot)$, where $\oplus : \mathbb{C}^n \times \mathbb{C}^n \in \mathbb{C}^n$ is given by $\oplus(z, w) = (z_1 + w_1, z_2 + w_2, \ldots, z_n + w_n)$ and $\odot : (\mathbb{C} \times \mathbb{C}^n) \cup (\mathbb{C}^n, \mathbb{C}) \in \mathbb{C}^n$ is given by $\odot(\alpha, z) = \odot(z, \alpha) = (\alpha z_1, \alpha z_2, \ldots, \alpha z_n)$. Then $\mathcal{C}^n$ is a complex vector space.
- Define $I : \mathbb{C}^n \times \mathbb{C}^n \in \mathbb{C}^n$ by $I(z, w) =$