Solving Brachistochrone Problem

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$$\int \sqrt{\frac{y}{c-y}}\ dy.......(\text{i})$$

Let:

$$u^2 = \frac{y}{c-y}......(\text{ii})$$
$$u^2c - u^2y = y$$
$$y + u^2y = u^2c$$
$$y(1+u^2) = u^2c$$
$$\therefore y = \frac{u^2c}{1+u^2}$$

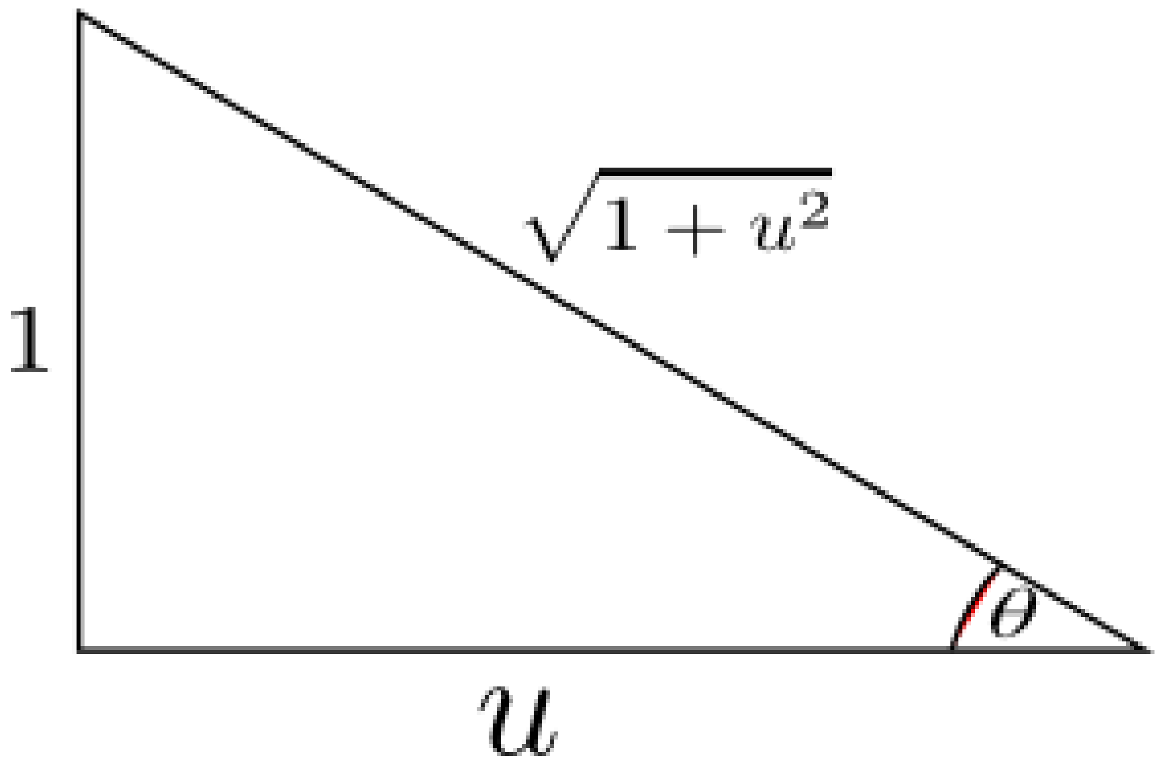
Now:

$$\frac{dy}{du} = \frac{(1+u^2)2uc - u^2c(2u)}{(1+u^2)^2}$$
$$= \frac{2uc + 2u^3c - 2u^3c}{(1+u^2)^2}$$
$$= \frac{2uc}{(1+u^2)^2}$$

$\therefore dy = \frac{2uc}{(1+u^2)^2}\ du......(\text{iii})$

Plugging in (ii) and (iii) into (i) we get:

$$2\int \frac{u^2c}{(1+u^2)^2}......(\text{iv})$$



Again let:

$$u = \cot(\theta)$$
$$du = -\csc^2(\theta)\ d\theta$$
$$\text{And } 1 + u^2 = 1 + \cot^2(\theta)$$

By plugging in all three of these values into (iv) we get:

$$
\begin{aligned}
&-2c\int \frac{\csc^2(\theta)\cot^2(\theta)}{\csc^4(\theta)}\ d\theta \\
&= -2c\int \frac{\cot^2(\theta)}{\csc^2(\theta)}\ d\theta \\
&= -c\int 2\cos^2(\theta)\ d\theta \\
&= -c\int (\cos(2\theta)+1)\ d\theta \\
&= -c\left(\frac{\sin(2\theta)}{2}+\theta\right)+C \\
&= -c\left(\sin(\theta)\cos(\theta)+\theta\right)+C \\
&= -c\left(\frac{u}{1+u^2}-\cot^{-1}\left(\sqrt{\frac{y}{c-y}}\right)\right)+C \\
&= \left(\sqrt{\frac{y}{c-y}}(y-c)+c\cot^{-1}\left(\sqrt{\frac{y}{c-y}}\right)\right)+C\text{...[Assuming both c and y > 0]}
\end{aligned}
$$

[Answer]