Derivation of the Lorentz Transformation in Einstein's Theory of Special Relativity

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One may consider two observers, S and S' with S' travelling at a velocity v relative to S, when an event occurs at an arbitrary location in line with v. S' travels in the direction towards the event, and the light from the event reaches S' over the distance x' during the course of time t' as measured by S'. One can therefore make a statement that

$$c = \frac{x'}{t'}$$

while the light from the event travels over distance x to S in the course of time t as measured by S. Thus, it can also be formulated that

$$c = \frac{x}{t}$$

These formulae can be rearranged to have

$$t = \frac{x}{c}$$

and

$$t' = \frac{x'}{c}$$

This leads to the conclusion that the measurements of time t by S and t' by S' are not equal. Considering that $x > x'$, it can be said that from the viewpoint of S,

$$x = x' + vt'$$

where vt' is the distance S' has travelled relative to S, while from the viewpoint of S',

$$x' = x - vt$$

where $-vt$ is the distance S has travelled relative to S'. Because it has been concluded the time quantities t and t' are not equivalent to each other, the two equations above contradict each other. To correct that, one must add a factor γ to modify the equations for them to equate each other. The altered pair of equations now becomes

$$x = \gamma\left(x' + vt'\right)$$

$$x' = \gamma\left(x - vt\right)$$

To solve for γ, the right and left sides of the equations are multiplied such that

$$xx' = \gamma^2\left(x' + vt'\right)\left(x - vt\right)$$

which simplifies to

$$xx' = \gamma^2\left(xx' - vtx' + vxt' - v^2tt'\right)$$

Undoing the earlier derived substitutions for t and t' gives

$$xx' = \gamma^2\left(xx' - \frac{vxx'}{c} + \frac{vxx'}{c} - \frac{v^2xx'}{c^2}\right)$$

Combine like terms,

$$xx' = \gamma^2\left(xx' - \frac{v^2xx'}{c^2}\right)$$

Divide both sides by xx',

$$1 = \gamma^2\left(1 - \frac{v^2}{c^2}\right)$$

Divide both sides by $\left(1 - \frac{v^2}{c^2}\right)$,

$$\frac{1}{1 - \frac{v^2}{c^2}} = \gamma^2$$

Take the square root of both sides,

$$\sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} = \gamma$$

And thus,

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This is known as the Lorentz factor for the Lorentz transformation in Einstein's Special relativity. It is the factor by which time, length, and relativistic mass change for a body moving relative to a frame of reference. It may also be formulated as

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

where β is the ratio of the velocity to the speed of light c, $\beta = \frac{v}{c}$. For instance, the relative change in time, also known as time dilation, of two observers can be modeled as

$$\Delta t' = \gamma \Delta t$$

Special relativity also describes the phenomenon of length contraction when a body is moving relative to a frame of reference. With Δx and $\Delta x'$ being the initial and final lengths of the body, respectively, length contraction is formulated as such,

$$\Delta x' = \frac{\Delta x}{\gamma}$$

Relativistic mass change describes the mass of a moving body changing depending on its velocity. Considering m_0 as the initial mass and m as the final mass of the body, relativistic mass change is shown as

$$m = \gamma m_0$$

These are several examples for which the Lorentz factor can be used in Special relativity, and the Lorentz transformation allows for further study of inertial frames of reference.