Midterm Exam - Takehome Part
Geometry - Spring 2015

This part of your test is due **Monday, March 16, 2015** Work each problem on a different sheet of paper, or use the tex template and typeset your work in LaTeX (you have the tex file of this test in D2L). Show all your work, including the explanations. Even when using the quantifiers, make sure your sentence make sense. Unsupported answers or messy work will receive no credit. Follow the grading rubrics for guidance.
You may only use your class notes, homework and the textbook to get help for solving the problems. It is not allowed to get help from any faculty member, any classmate, or any other person. Please refer to the academic honesty policy on the course syllabus.

Choose four problem to solve for credit. Only the problems you choose will be graded. The fifth problem may be submitted after the spring break as a bonus if you want to.

1. Given a line segment and an angle construct using a collapsible compass and a straightedge an isosceles triangle having the base congruent to the given segment and the base angles congruent to the given angle. (suppose the segment does not lie on any of the two sides of the angle and the vertex of the angle does not lie on the line containing the segment.)

2. Use the order on $\mathbb{Q}$ to define the betweenness axioms in $\mathbb{Q}^2$ and prove one of the betweenness proposition (of your choice).

3. **(a)** Show that in any affine plane the parallelism relation is transitive, that is, $\forall l\ \forall m\ \forall n\ (l \| m\ \&\ m \| n\ \&\ l \neq n \Rightarrow l \| n)$. Explain why should $l \neq n$ be assumed.
   (b) Show that if a model has this property it must be an affine plane.
   (c) Considering what you proved in the first two parts and the examples we discussed in class think about a model of incidence geometry in which parallel lines exist but parallelism is not transitive.

4. A set of points $\mathcal{S}$ is called *convex* if whenever two points P and Q are in $\mathcal{S}$, the entire segment PQ is contained in $\mathcal{S}$. Prove that the interior of a triangle $\triangle ABC$ is a convex set.

5. Prove that in an equilateral triangle $\triangle ABC$
   (a) all angles are congruent to each other;
   (b) all three angle bisector segments are congruent to each other.

1. Follow attached file for diagram.
Start with points A and B on a line. Construct an angle with line l for duplication off of A. Pick a point C to start the construction of the isosceles triangle. Construct an equilateral triangle ACD. Construct a circle from point A with radius $\overline{AB}$. Extend line AD to that circle. Point E represents the intersection of line AD to the circle. Draw circle from point D with radius $\overline{DE}$. Extend line DC to that circle, with point F being the intersection. draw a circle from point F with a radius larger than the perpendicular of F to line l. This will give you two points to construct a perpendicular off of l through F. Then construct a perpendicular to that perpendicular line running through F in order to have a parallel of l through F. Angle from this parallel and $\overline{CF}$ is a duplication of the given angle. Construct a perpendicular of $\overline{CF}$ through the midpoint. Point I is at the intersection of this perpendicular and the parallel line constructed through F. Connect points C and I. $\triangle CIF$ is an isosceles triangle with the given line segment $\overline{AB}$ translated by $\overline{CF}$ and the given angle translated by $\angle FCI$ and $\angle ICF$.

2. SKIPPING NUMBER 2

3. **(a)** Hypothesis: $\forall l \forall m \forall n$, $l \parallel m$ & $m \parallel n$ & $l \neq n$
   Conclusion: $l \parallel n$
   Proof: First off, we must assume $l \neq n$ because it is possible for $l \parallel m$ & $m \parallel n$, but if $l = n$ then l and n intersect at every point and are not parallel. Thus l must not equal n. Assume l is not parallel to n. Then n and l share a common point of intersect. Since $l \parallel m$ then n will eventually traverse through m. Thus n and m would also share a point of common intersect. This contradicts that $m \parallel n$. Then by RAA, if $l \parallel m$, $m \parallel n$ and $l \neq n$ then $l \parallel n$.

   (b) Hypothesis: Model holds transitive parallelism relation.
   Conclusion: Model is an affine plane.
   Proof: If the model has transitive parallelism then clearly we know the parallel postulate holds. In this geometry we can also see that the incidence axioms must hold. If we take two points we must be able to find a unique line between them. Likewise, for us to hold parallel lines we must be able to define two points on those lines. The last incidence axiom holds, otherwise parallel lines could not exist at all. Thus if this said model holds the parallel postulate and the incidence axioms then it is an affine plane.

   (c) A model in which this occurs can be where a line is represented any circle around a sphere. The circles do not have to measure the same diameter. In one sense, imagine any ring, of any size, that you could lay down on Earth. Two rings that intersect on part of the sphere may not share any points in common with a third ring. Thus, $1 \parallel 2$, $2 \parallel 3$, but 1 is not parallel to 3.

4. Hypothesis: $\exists \triangle ABC$
   Conclusion: interior of $\triangle ABC$ is a convex set.
   Proof: Let I be the set of points in the interior of $\triangle ABC$. Take two points $P, Q \in I$. Observe $\angle PAQ$. By Betweenness Proposition 0.8, we know that $\forall R$, such that $P * R * Q$, R is in the interior of $\angle PAQ$. To avoid redundancy, we can see the same is true for $\angle PBQ$ and $\angle PCQ$. Thus $\forall R$, such that $P * R * Q$, R is in the interior of $\triangle ABC$. Since we already know P and Q are in the interior of $\triangle ABC$, then the entire segment $\overline{PQ}$ is contained in I. Therefore, the interior of $\triangle ABC$ is a convex set.

5. **(a)** Hypothesis: $\exists \triangle ABC$
   Conclusion: $\angle A \cong \angle B \cong \angle C$
   Proof: First we take the midpoint of a side, say $\overline{BC}$, and call it point D. Also we connect the points A and D with a line segment. Next we can observe that $\overline{AB} \cong \overline{AC}$, since all lines are congruent, also $\overline{AD} \cong \overline{AD}$ by Congruence Axiom 2, and lastly $\overline{BD} \cong \overline{DC}$, since midpoint D divides $\overline{BC}$ into two equal parts. Then by SSS congruence $\triangle BDA \cong \triangle CDA$. Thus $\angle B \cong \angle C$. For prevention of redundancy, we can conclude the same about $\angle A$ and $\angle C$ if we take a midpoint E on $\overline{AC}$. Then we have $\angle B \cong \angle C$ and $\angle A \cong \angle C$. Then by transitivity of congruent angles, $\angle A \cong \angle B \cong \angle C$. Therefore, all angles are congruent to each other.

   (b) Hypothesis: $\exists \triangle ABC$
   Conclusion: $\overline{AD} \cong \overline{BE} \cong \overline{CF}$, with $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ being the three angle bisectors (Note: in an equilateral triangle the angle bisector are the same as the medians).
   Proof: We will compare the following three triangles: $\triangle ADC$, $\triangle BEA$ and $\triangle CFB$. Next we can observe that since all sides are equal, then 1/2 of any side must also be equal. This means that $\overline{DC} \cong \overline{EA} \cong \overline{FB}$ for they all represent 1/2 of the divided lines by their respective midpoints. Also, as proved in part (a), $\angle C \cong \angle A \cong \angle B$. Since, $\triangle ABC$ is equilateral, we already know $\overline{CA} \cong \overline{AB} \cong \overline{BC}$. Then by SAS congruence, these three triangles are congruent. Therefore, $\overline{AD} \cong \overline{BE} \cong \overline{CF}$.