Griffith QM Time Dependent Perturbation Theory CheatSheet (UCB 137B)

TIPT

$$H = H_0 + H'$$
$$E_n = E_n^0 + E_n^1$$
$$|\psi_n\rangle = |\psi_n^0\rangle + |\psi_n^1\rangle$$
$$E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle$$
$$|\psi_n^1\rangle = \sum_{m\neq n} \frac{\langle\psi_m^0|H'|\psi_n^0\rangle}{E_n^0 - E_m^0}|\psi_m^0\rangle$$

Degenerate Case

$$\text{Degenerate space: } \{|i\rangle\} \to E$$
$$W_{ab} = \langle a|H'|b\rangle \text{ Non-Diagonal}$$
$$\text{Eigenvalue and Eivenvectors} \to E_n^1, |\hat{i}\rangle$$

Variational Method

$$\langle H\rangle(\lambda) = \frac{\langle\psi(x,\lambda)|H|\psi(x,\lambda)\rangle}{\langle\psi(x,\lambda)|\psi(x,\lambda)\rangle}$$
$$\langle H\rangle(\lambda) \geq E_{.g.s}$$
$$\frac{d}{d\lambda}\langle H\rangle(\lambda_0) = 0 \Rightarrow \langle H\rangle(\lambda_0) \approx E_{.g.s}$$

WKB Method

$$\frac{d^2\psi(x)}{dx^2} = -k^2(x)\psi(x)$$
$$k(x) = \frac{1}{\hbar}\sqrt{2m(E-V(x)}$$
$$\phi(x) = \int^x k(x)dx$$
$$\psi(x) = \frac{1}{\sqrt{k(x)}}(C_+e^{i\phi(x)} + C_-e^{-i\phi(x)})$$
$$= \frac{1}{\sqrt{k(x)}}(C_1\sin\phi(x) + C_2\cos\phi(x))$$

Energy Level

$$\int_{R_{classical}} k(x)dx = n\pi$$
$$\text{one } \infty \text{ wall} \quad n \to n - 1/4$$
$$\text{No } \infty \text{ wall} \quad n \to n - 1/2$$

Tunneling

$$T = e^{-2\gamma}$$
$$\gamma = \int_{R_{forbidden}} k(x)dx$$

TDPT

$$H = H_0 + V(t)$$
$$\text{Eigenstate of } H_0\text{: } |n\rangle, E_n$$
$$\text{transition: } |i\rangle \to |f\rangle$$
$$V_{fi}(t) = \langle f|V(t)|i\rangle$$
$$\omega_{fi} = (E_f - E_i)/\hbar$$
$$c_f(T) = \frac{-i}{\hbar}\int_0^T V_{fi}(t)e^{-i\omega_{fi}t}dt$$

Constant Perturbation

$$V(t) = \begin{cases} V, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$
$$V_{fi}(t) = constant$$
$$P_{i\to f}(t) = |c_f(t)|^2 = 4\frac{|V_{fi}|^2}{\hbar^2}\frac{\sin^2(\omega_{fi})t/2}{\omega_{fi}^2}$$
$$\omega_{fi} \to 0 \quad \text{(degenerate states):}$$
$$|c_f(t)|^2 = \frac{|V_{fi}|^2}{\hbar^2}t^2$$

Absorption

$$V(t) = V\sin(\omega t)$$
$$P_{i\to f}(t) = \frac{|V_{fi}|^2}{\hbar^2}\frac{\sin^2((\omega_{fi}-\omega)t/2)}{(\omega_{fi}-\omega)^2}$$

Simulated Emission

$$E_i > E_f, \quad \omega_{fi} < 0$$
$$P_{i\to f}(t) = \frac{|V_{fi}|^2}{\hbar^2}\frac{\sin^2((\omega_{fi}+\omega)t/2)}{(\omega_{fi}+\omega)^2}$$

Fermi Golden Rule

$$E_i \to E_f \text{ (continuous states)}$$
$$P_{i\to f} = \frac{2\pi}{\hbar}|\langle f|V|i\rangle|^2\rho(E_f)t$$

Selection Rule

For spherical symmetric potential:

$$\langle n', l', m'|\vec{r}|n, l, m\rangle \neq 0 \text{ when:}$$
$$\Delta l = \pm 1 \text{ and:}$$
$$\Delta l = \pm 1 \text{ or } 0$$

Scattering

$$\psi(r,\theta) = e^{ikz} + f(\theta)\frac{e^{ikr}}{r}, \text{ for large r}$$
$$k = \frac{\sqrt{2mE}}{\hbar}$$
$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$
$$\sigma = \int d\omega\frac{d\sigma}{d\Omega}$$

Born Approximation

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int V(\vec{r})e^{i(\vec{k}'-\vec{k})\cdot\vec{r}}d^3\vec{r}$$

Low Energy:

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int V(\vec{r})d^3\vec{r}$$

Spherical symmetric:

$$f(\theta) = -\frac{2m}{\hbar^2\kappa}\int_0^\infty rV(r)\sin(\kappa r)dr$$
$$\kappa = 2k\sin(\theta/2)$$

Yukawa Potential

$$V(r) = V_0\frac{e^{-r/R}}{r}$$
$$f(\theta) = -\frac{2mV_0R^2}{\hbar^2}\frac{1}{1+4k^2R^2\sin^2(\theta/2)}$$
$$\sigma = (\frac{2mV_0R^2}{\hbar^2})^2\frac{4\pi}{1+4k^2R^2}$$

Rutherford Scattering

Let $V_0 = q_1q_2/4\pi\epsilon_0$, $R = \infty$:

$$f(\theta) = -\frac{2mq_1q_2}{4\pi\epsilon_0\hbar^2\kappa^2}$$

Partial Waves

$$f(\theta) = \frac{1}{k}\sum_{i=0}^\infty(2l+1)e^{i\delta_l}\sin(\delta_l)P_l(\cos(\theta))$$
$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^\infty(2l+1)\sin^2(\delta_l)$$

Optical Theorem

$$Im[f(0)] = \frac{k\sigma}{4\pi}$$

Hard Ball

$$\delta_l = \tan^{-1}(\frac{j_l(ka)}{\eta_l(ka)})$$
$$ka << 1 \to \sigma = 4\pi a^2$$

Useful Models

Density of States

$$E = \hbar^2 k^2/2m$$
$$dN = \frac{L^3}{(2\pi)^3} d^3k = \frac{L^3}{(2\pi)^3} d\Omega dk$$
$$dN = \frac{L^3}{(2\pi)^3} 4\pi \frac{m}{\hbar^2 k} dE$$
$$\rho(E) = \frac{dN}{dE} = \frac{L^3}{2\pi^2}\frac{mk}{\hbar^2}$$

infinite square well

$$H(x) = \frac{p^2}{2m} + \begin{cases} 0, & 0 \le x \le a \\ \infty, & \text{otherwise} \end{cases}$$
$$E_n = \frac{1}{2m}(\frac{n\pi\hbar}{a})^2$$
$$\psi_n = \sqrt{\frac{2}{a}} \sin(\frac{n\pi x}{a}) e^{-iE_n t/\hbar}$$

Harmonic Oscillator

$$H(x) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$
$$E_n = (n+1/2)\hbar\omega$$
$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}(\frac{m\omega}{\pi\hbar})^{1/4} e^{-\zeta^2/2} H_n(\zeta)$$
$$\zeta = \sqrt{\frac{m\omega}{\hbar}} x$$

Virial Theorem

$$2\langle T\rangle = \langle \vec{r}\cdot\nabla V\rangle \quad (3D)$$
$$2\langle T\rangle = \langle x\frac{dV}{dx}\rangle \quad (1D)$$
$$2\langle T\rangle = n\langle V\rangle \quad (V \propto r^n)$$
$$\langle T\rangle = -E_n, \quad \langle V\rangle = 2E_n \quad \text{(hydrogen)}$$
$$\langle T\rangle = \langle V\rangle = E_n/2 \quad \text{(harmonic oscillator)}$$

Math

Legendre Polynomials

Domain: $(-1, 1)$
Even, Odd, Even, Odd ...

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Hankel Functions

Solution to Radial Shrodinger Equation:

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 R_{El}) + [V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}]R_{El} = ER_{El}$$
$$V = 0 \to R_{El} = j_l(kr)$$
$$V \neq 0 \to R_{El} = j_l(kr + \delta_l)$$
$$r \to \infty \Rightarrow R_{El} = \frac{\sin(kr - l\pi/2 + \delta_l(E))}{kr}$$

When $kr >> 1$

$$j_l(kr) \to \frac{\sin kr - l\pi/2}{kr}$$
$$\eta_l(kr) \to \frac{-\cos kr - l\pi/2}{kr}$$
$$h_l(kr) \to \frac{e^{i(kr - l\pi/2)}}{ikr}$$
$$h_l^*(kr) \to \frac{e^{-i(kr - l\pi/2)}}{-ikr}$$
$$j_l(kr) = \frac{1}{2}(h_l(kr) + h^*(kr))$$

Hermite Polynomials

Domain: $(-\infty, \infty)$
Even, Odd, Even, Odd ...

$$H_0(x) = 1$$
$$H_1(x) = 2x$$
$$H_2(x) = 4x^2 - 2$$
$$H_3(x) = 8x^3 - 12x$$

Spherical Harmonics

$$|l, m\rangle = Y_l^m(\theta, \phi)$$
$$Y_0^0(\theta, \phi) = \frac{1}{2}\frac{1}{\sqrt{\pi}}$$
$$Y_1^0(\theta, \phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$$
$$Y_1^{-1}(\theta, \phi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\phi}$$
$$Y_1^{-1}(\theta, \phi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{i\phi}$$

Green's Function

For a Linear Operator $\hat{D}_x$

$$\text{Homogeneous solution: } \hat{D}_x \psi_0(x) = 0$$
$$\text{Hard Problem: } \hat{D}_x \psi(x) = f(x)$$
$$\text{Simple Problem: } \hat{D}_x G(x, x') = \delta(x - x')$$
$$\psi(x) = \psi_0(x) + \int_{\text{f Domain}} G(x, x') f(x') dx'$$

Some Integrals

$$\Gamma(n+1) = n!$$
$$\Gamma(z+1) = z\Gamma(z)$$
$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$
$$\int_0^\infty e^{-ax^b} dx = a^{-1/b}\Gamma(1/b + 1)$$
$$\int_0^\infty e^{-ax}\sin bx dx = \frac{b}{a^2 + b^2}$$
$$\int_0^\infty e^{-ax}\cos bx dx = \frac{a}{a^2 + b^2}$$
$$\int_{-\infty}^\infty e^{-ax^2 + bx} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$$
$$\int_0^\infty e^{-ax^2} x^n dx = I_n(a)$$
$$I_0 = \frac{1}{2}\sqrt{\frac{\pi}{a}}, I_1 = \frac{1}{2a}, I_2 = \frac{1}{4a}\sqrt{\frac{\pi}{a}}, I_3 = \frac{1}{2a^2}$$