Involutory Property of the Discrete Hartley Transform

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Given a column vector $x \in \mathbb{R}^n$, its *Discrete Hartley Transform* (DHT) is defined as another vector $y \in \mathbb{R}^n$ such that

$$y_j = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} x_i \operatorname{cas}\left(\tfrac{2\pi}{n} ij\right) \quad \text{for} \quad j \in \{0, \cdots, n-1\} \tag{1}$$

where the cas function is defined as $\operatorname{cas} \vartheta = \cos \vartheta + \sin \vartheta$. Interestingly, the DHT is an *involution*; that is, the DHT is the same as the inverse DHT.

$$x_i = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} y_j \operatorname{cas}\left(\tfrac{2\pi}{n} ij\right) \quad \text{for} \quad i \in \{0, \cdots, n-1\} \tag{2}$$

This paper proves the DHT is indeed an involution.

Target Equality To simplify (1) and (2), define an $n \times n$ symmetric matrix H whose (i, j)-entry is $\frac{1}{\sqrt{n}} \operatorname{cas}\left(\frac{2\pi}{n} ij\right)$. Then the DHT and the inverse DHT are

$$y = Hx \quad \text{and} \quad x = Hy$$

where $x = \left[x_0, \cdots, x_{n-1}\right]^\intercal$ and $y = \left[y_0, \cdots, y_{n-1}\right]^\intercal$. Then proving the involutory property of the DHT reduces to showing that $H^2 = I$ where I is an $n \times n$ identity matrix. This further reduces to showing that the rows in H are orthogonal; that is,

$$\langle h_i, h_{i'} \rangle = \begin{cases} 1, & \text{if } i = i'; \\ 0, & \text{otherwise;} \end{cases}$$

where $h_i = \frac{1}{\sqrt{n}} \left[\operatorname{cas}\left(\frac{2\pi}{n} i0\right), \cdots, \operatorname{cas}\left(\frac{2\pi}{n} i(n-1)\right)\right]^\intercal$ is the i^{th} row in H. It may be written in terms of cas functions as

$$\sum_{j=0}^{n-1} \operatorname{cas}\left(\tfrac{2\pi}{n} ij\right) \operatorname{cas}\left(\tfrac{2\pi}{n} i'j\right) = \begin{cases} n, & \text{if } i = i'; \\ 0, & \text{otherwise;} \end{cases} \tag{3}$$

which is the target equality for the involutory property.

CAS Identity The proof of (3) begins with an identity about cas functions.

$$\operatorname{cas}\alpha \operatorname{cas}\beta = \sin(\alpha+\beta) + \cos(\alpha-\beta) \tag{4}$$

Proof of (4). A $\operatorname{cas}(\cdot)$ can be simplified to $\cos(\cdot)$ as follows:

$$\operatorname{cas}\vartheta = \cos\vartheta + \sin\vartheta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\vartheta + \frac{1}{\sqrt{2}}\sin\vartheta\right) = \sqrt{2}\cos\left(\vartheta - \frac{\pi}{4}\right) \tag{$*$}$$

Then

$$\begin{aligned}
\operatorname{cas}\alpha \operatorname{cas}\beta &= \sqrt{2}\cos\left(\alpha-\frac{\pi}{4}\right)\cdot\sqrt{2}\cos\left(\beta-\frac{\pi}{4}\right) && \text{by } (*) \\
&= \cos\left(\alpha+\beta-\frac{\pi}{2}\right) + \cos(\alpha-\beta) && 2\cos\vartheta\cos\phi = \cos(\vartheta+\phi)+\cos(\vartheta-\phi) \\
&= \sin(\alpha+\beta) + \cos(\alpha-\beta) && \cos\left(\vartheta-\frac{\pi}{2}\right) = \sin\vartheta
\end{aligned}$$

which completes the proof. □

Target Simplified The target equality (3) is decomposed into two summations by (4).

$$\sum_{j=0}^{n-1} \operatorname{cas}\left(\tfrac{2\pi}{n} ij\right) \operatorname{cas}\left(\tfrac{2\pi}{n} i'j\right) = \sum_{j=0}^{n-1} \sin\left(\tfrac{2\pi}{n}(i+i')j\right) + \sum_{j=0}^{n-1} \cos\left(\tfrac{2\pi}{n}(i-i')j\right)$$

The above will be interpreted as the real and imaginary parts in geometric progressions of complex numbers:

$$\sum_{j=0}^{n-1} \operatorname{cas}\left(\tfrac{2\pi}{n} ij\right) \operatorname{cas}\left(\tfrac{2\pi}{n} i'j\right) = \Im\left\{\sum_{j=0}^{n-1} \omega^{(i+i')j}\right\} + \Re\left\{\sum_{j=0}^{n-1} \omega^{(i-i')j}\right\} \tag{5}$$

where ω is the *primitive* n^{th} *root of unity* $\omega = \exp(\imath 2\pi/n) = \cos(2\pi/n) + \imath\sin(2\pi/n)$. The identity (5) is easily proved by the De Moivre's identity.

Summation Lemma The last puzzle to the involutory property proof is the summation of a geometric series:

$$\text{For an integer } k, \quad \sum_{j=0}^{n-1} (\omega^k)^j = \begin{cases} n, & \text{if } k \text{ is a multiple of } n; \\ 0, & \text{otherwise.} \end{cases}$$

This lemma ensures that the imaginary part of $\sum_{j=0}^{n-1} (\omega^k)^j$ is zero regardless of the integer value k. On the other hand, the real part is n if k is a multiple of n and zero otherwise. Plugging in theses values to the RHS of (5) yields the desired identity (3), which completes the involutory property proof.