Modeling Walk Efficiency in a City Grid With Markov Processes

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Table of contents

Introduction

The Markov Property

We say that a sequence $\{X_n, n \in T\}$, where T is a subset of the integers, has the **Markov Property** if,

for any $n \in T$,
the future process $\{X_{n+1}\}$
is dependent only on the current state $\{X_n\}$
and is independent of past processes $\{X_i\}$ for integers $0 \leq i < n$.

Notation

Elements of the state space, S, are commonly denoted as $(i_0, i_1, ..., i_n)$ that is to say that

$$X_n = i_n \in S$$

Notation cont.

The **transition probability** is given by the function

$$p_{i,j}(n, n+1) := P(X_{n+1} = j | X_n = i)$$

where the transition is from state i at time n to state j at time $n+1$. From this definition we can define the probability matrix as

$$P(n, n+1) := [p_{i,j}(n, n+1)]_{i,j \in S}$$

Notation cont.

A Markov process in which the probability distribution does not depend on time is called **time-homogeneous** and its probability matrix can simple be defined simply as

$$P := [p_{i,j}]_{i,j \in S}$$

An example of a probability matrix for a time-homogeneous system with three states could be

$$P = \begin{pmatrix} 0 & 0.3 & 0.7 \\ 0.15 & 0.45 & 0.4 \\ 0 & 1 & 0 \end{pmatrix}$$

*Note: each row must add up to 1 to represent the probability that the system must transition from a particular state to another state in the system. We see in this example that a state is allowed to transition to itself or to transition to another given state with probability 1.

Notation cont.

Given the probability matrix P, we can determine the probability distribution for transitioning from state i to state j in n steps by taking our matrix to the *nth* power. Thus, in the matrix P^n, we have that

$$P^n := [p^n_{i,j}]_{i,j \in S}$$

where **each entry of the matrix correlates to the probability of transitioning from the ith state to the jth state in n steps**.

My Project

The City Grid

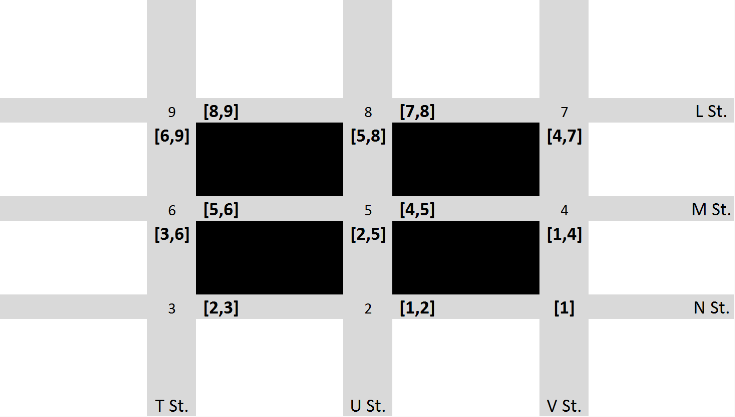

Figure 1: A city grid including states in the state space

Paths Through the City

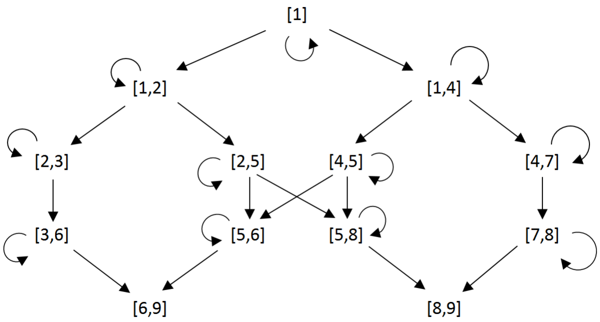

Figure 2: Our 6 possible paths (from top to bottom) through the grid

Building the Transition Matrix

		{1}	{1,2}	{1,4}	{2,5}	{2,3}	{4,5}	{4,7}	{3,6}	{5,6}	{5,8}	{7,8}	{6,9}	{8,9}
	{1}	0	0.5	0.5	0	0	0	0	0	0	0	0	0	0
	{1,2}	0	0	0	0.5	0.5	0	0	0	0	0	0	0	0
	{1,4}	0	0	0	0	0	0.5	0.5	0	0	0	0	0	0
	{2,5}	0	0	0	0	0	0	0	0	0.5	0.5	0	0	0
	{2,3}	0	0	0	0	0	0	0	1	0	0	0	0	0
	{4,5}	0	0	0	0	0	0	0	0	0.5	0.5	0	0	0
	{4,7}	0	0	0	0	0	0	0	0	0	0	1	0	0
P =	{3,6}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{5,6}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{5,8}	0	0	0	0	0	0	0	0	0	0	0	0	1
	{7,8}	0	0	0	0	0	0	0	0	0	0	0	0	1
	{6,9}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{8,9}	0	0	0	0	0	0	0	0	0	0	0	0	1

Figure 3: The transition matrix for all green lights and no preference for any direction

Building the Transition Matrix

$$p_{i,j} = \begin{cases} p + myfav & \text{if continuing straight} \\ p - myfav & \text{if turning} \end{cases}$$

*Note: The previous matrix had *myfav* value of 0. We will now set $myfav = 0.3$ to represent a strong preference to continue straight.

Building the Transition Matrix

		{1}	{1,2}	{1,4}	{2,5}	{2,3}	{4,5}	{4,7}	{3,6}	{5,6}	{5,8}	{7,8}	{6,9}	{8,9}
	{1}	0	0.5	0.5	0	0	0	0	0	0	0	0	0	0
	{1,2}	0	0	0	0.2	0.8	0	0	0	0	0	0	0	0
	{1,4}	0	0	0	0	0	0.2	0.8	0	0	0	0	0	0
	{2,5}	0	0	0	0	0	0	0	0	0.2	0.8	0	0	0
	{2,3}	0	0	0	0	0	0	0	1	0	0	0	0	0
	{4,5}	0	0	0	0	0	0	0	0	0.8	0.2	0	0	0
	{4,7}	0	0	0	0	0	0	0	0	0	0	1	0	0
P =	{3,6}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{5,6}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{5,8}	0	0	0	0	0	0	0	0	0	0	0	0	1
	{7,8}	0	0	0	0	0	0	0	0	0	0	0	0	1
	{6,9}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{8,9}	0	0	0	0	0	0	0	0	0	0	0	0	1

Figure 4: The transition matrix for all green lights and strong preference for continuing straight

Building the Transition Matrix

Next, I want to add the possibility that the light in one direction is red, this will rely on traffic variables that I've assigned to each street.

The equation for each cell will be as follows:

$$= IF(G2 = 0, 0, IF(SUM(\$F2 : F2) = 0, MAX(G2, traffic), 1 - SUM(\$F18 : F18)))$$

Essentially, determines if the light is green in the preferred direction and if so will take the maximum value between the previous cell value and the probability of a green light.

Building the Transition Matrix

		{1}	{1,2}	{1,4}	{2,5}	{2,3}	{4,5}	{4,7}	{3,6}	{5,6}	{5,8}	{7,8}	{6,9}	{8,9}
	{1}	0	0.5	0.5	0	0	0	0	0	0	0	0	0	0
	{1,2}	0	0	0	0.5	0.5	0	0	0	0	0	0	0	0
	{1,4}	0	0	0	0	0	0.5	0.5	0	0	0	0	0	0
	{2,5}	0	0	0	0	0	0	0	0	0.85	0.15	0	0	0
	{2,3}	0	0	0	0	0	0	0	1	0	0	0	0	0
	{4,5}	0	0	0	0	0	0	0	0	0.85	0.15	0	0	0
	{4,7}	0	0	0	0	0	0	0	0	0	0	1	0	0
P=	{3,6}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{5,6}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{5,8}	0	0	0	0	0	0	0	0	0	0	0	0	1
	{7,8}	0	0	0	0	0	0	0	0	0	0	0	0	1
	{6,9}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{8,9}	0	0	0	0	0	0	0	0	0	0	0	0	1

Figure 5: The transition matrix for choosing a direction with traffic

Building the Transition Matrix

		{1}	{1,2}	{1,4}	{2,5}	{2,3}	{4,5}	{4,7}	{3,6}	{5,6}	{5,8}	{7,8}	{6,9}	{8,9}
	{1}	0.05	0.475	0.475	0	0	0	0	0	0	0	0	0	0
	{1,2}	0	0.2	0	0.2	0.6	0	0	0	0	0	0	0	0
	{1,4}	0	0	0.25	0	0	0.175	0.575	0	0	0	0	0	0
	{2,5}	0	0	0	0.35	0	0	0	0	0.65	0	0	0	0
	{2,3}	0	0	0	0	0	0	0	1	0	0	0	0	0
	{4,5}	0	0	0	0	0	0.35	0	0	0.65	0	0	0	0
	{4,7}	0	0	0	0	0	0	0.15	0	0	0	0.85	0	0
P=	{3,6}	0	0	0	0	0	0	0	0.1	0	0	0	0.9	0
	{5,6}	0	0	0	0	0	0	0	0	0.2	0	0	0.8	0
	{5,8}	0	0	0	0	0	0	0	0	0	0.1	0	0	0.9
	{7,8}	0	0	0	0	0	0	0	0	0	0	0.35	0	0.65
	{6,9}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{8,9}	0	0	0	0	0	0	0	0	0	0	0	0	1

Figure 6: The transition matrix, P, for choosing a direction with traffic and red lights

Calculations

Two Approaches

- Worst Case Scenario (Maximum Travel Time)
    - $$p^k_{\{1\},\{6,9\}} + p^k_{\{1\},\{8,9\}} \geq 1 - \epsilon, \epsilon < 0.0001$$
- Expected Time
    - $$\text{Expected Time} = \sum_{k=4}^{14} k[p^k_{\{1\},\{6,9\}} + p^k_{\{1\},\{8,9\}}]$$

Worst Case Scenario

		{1}	{1,2}	{1,4}	{2,5}	{2,3}	{4,5}	{4,7}	{3,6}	{5,6}	{5,8}	{7,8}	{6,9}	{8,9}
	{1}	0	0.005	0.0093	0.0249	0.015	0.0254	0.0396	0.0997	0.0953	0	0.1857	0.3491	0.1509
	{1,2}	0	0.0016	0	0.0179	0.0048	0	0	0.042	0.0497	0	0	0.884	0
	{1,4}	0	0	0.0039	0	0	0.0194	0.0195	0	0.0492	0	0.1882	0.1638	0.556
	{2,5}	0	0	0	0.015	0	0	0	0	0.0581	0	0	0.9269	0
	{2,3}	0	0	0	0	0	0	0	0.001	0	0	0	0.999	0
	{4,5}	0	0	0	0	0	0.015	0	0	0.0581	0	0	0.9269	0
	{4,7}	0	0	0	0	0	0	0.0005	0	0	0	0.0616	0	0.9379
P^4=	{3,6}	0	0	0	0	0	0	0	0.0001	0	0	0	0.9999	0
	{5,6}	0	0	0	0	0	0	0	0	0.0016	0	0	0.9984	0
	{5,8}	0	0	0	0	0	0	0	0	0	0.0001	0	0	0.9999
	{7,8}	0	0	0	0	0	0	0	0	0	0	0.015	0	0.985
	{6,9}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{8,9}	0	0	0	0	0	0	0	0	0	0	0	0	1

Figure 7: Our transition matrix, P, from Figure 6 after 4 steps

Worst Case Scenario

		{1}	{1,2}	{1,4}	{2,5}	{2,3}	{4,5}	{4,7}	{3,6}	{5,6}	{5,8}	{7,8}	{6,9}	{8,9}
	{1}	0	0	0	0	0	0	0	0	0	0	0	0.6167	0.3833
	{1,2}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{1,4}	0	0	0	0	0	0	0	0	0	0	0	0.2333	0.7667
	{2,5}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{2,3}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{4,5}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{4,7}	0	0	0	0	0	0	0	0	0	0	0	0	1
P^14=	{3,6}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{5,6}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{5,8}	0	0	0	0	0	0	0	0	0	0	0	0	1
	{7,8}	0	0	0	0	0	0	0	0	0	0	0	0	1
	{6,9}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{8,9}	0	0	0	0	0	0	0	0	0	0	0	0	1

Figure 8: Our transition matrix, P, from Figure 6 after 14 steps

Worst Case Scenario

		{1}	{1,2}	{1,4}	{2,5}	{2,3}	{4,5}	{4,7}	{3,6}	{5,6}	{5,8}	{7,8}	{6,9}	{8,9}
	{1}	0	0	0	0	0	0	0	0	0	0	0	0.75	0.25
	{1,2}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{1,4}	0	0	0	0	0	0	0	0	0	0	0	0.5	0.5
	{2,5}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{2,3}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{4,5}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{4,7}	0	0	0	0	0	0	0	0	0	0	0	0	1
A^14=	{3,6}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{5,6}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{5,8}	0	0	0	0	0	0	0	0	0	0	0	0	1
	{7,8}	0	0	0	0	0	0	0	0	0	0	0	0	1
	{6,9}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{8,9}	0	0	0	0	0	0	0	0	0	0	0	0	1

Figure 9: Matrix A with $myfav = 0$ after 14 steps

Worst Case Scenario

		{1}	{1,2}	{1,4}	{2,5}	{2,3}	{4,5}	{4,7}	{3,6}	{5,6}	{5,8}	{7,8}	{6,9}	{8,9}
	{1}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{1,2}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{1,4}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{2,5}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{2,3}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{4,5}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{4,7}	0	0	0	0	0	0	0	0	0	0	0	0	1
B^14=	{3,6}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{5,6}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{5,8}	0	0	0	0	0	0	0	0	0	0	0	0	1
	{7,8}	0	0	0	0	0	0	0	0	0	0	0	0	1
	{6,9}	0	0	0	0	0	0	0	0	0	0	0	1	0
	{8,9}	0	0	0	0	0	0	0	0	0	0	0	0	1

Figure 10: Matrix B with $myfav = -0.3$ after 14 steps

Expected Time

$$\text{Expected Time} = \sum_k k(p_{1,j}^k)$$

where $p_{1,j}^k$ is the probability of reaching the destination after k steps.

$$\text{Expected Time} = \sum_{k=4}^{14} k[p_{\{1\},\{6,9\}}^k + p_{\{1\},\{8,9\}}^k]$$

Expected Time

Results are forthcoming…