On Proving Leonhard Euler's Evaluation of the Riemann Zeta Function of 2

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Let one consider the Riemann zeta function $\zeta(z)$, with a given integral identity,

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{x^{z-1}}{e^x - 1}\, dx$$

In the case where $z = 2$, one can evaluate the zeta function of 2 by substituting 2 for z in the integrand,

$$\zeta(2) = \frac{1}{\Gamma(2)} \int_0^\infty \frac{x^{2-1}}{e^x - 1}\, dx$$

which simplifies to

$$\zeta(2) = \int_0^\infty \frac{x}{e^x - 1}\, dx$$

To evaluate the integral, the property of geometric series can be shown,

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1$$

The denominator of the integrand is the term $e^x - 1$. In order to turn the e^x term to 1 and the -1 term to something else, it can be multiplied by e^{-x}. However, the numerator x of the integrand has to be multiplied by e^{-x} as well in order for the integral expressions to be equal. Thus one has

$$\int_0^\infty \frac{x}{e^x - 1}\, dx = \int_0^\infty \frac{xe^{-x}}{1 - e^{-x}}\, dx$$

Looking back to the property of geometric series, a can be set as $a = xe^{-x}$, and r as $r = e^{-x}$. And because $|e^{-x}| < 1$ from 0 to infinity, it allows to do the following,

$$\int_0^\infty \frac{x}{e^x - 1}\, dx = \int_0^\infty \sum_{n=1}^{\infty} xe^{-x}e^{-xn+x}\, dx$$

which simplifies to

$$\int_0^\infty \frac{x}{e^x - 1}\, dx = \int_0^\infty \sum_{n=1}^{\infty} xe^{-xn}\, dx$$

In this power series, it can be stated that e^{-xn} will decrease at a much greater rate than x will increase, and thus for any real number x the infinite series

$$\sum_{n=1}^{\infty} xe^{-xn}$$

will uniformly converge. Taking this into account with the derived integral, one is allowed to switch the integral and summation signs under the condition that the series has a pointwise convergence,

$$\int_0^{\infty} \sum_{n=1}^{\infty} xe^{-xn}\, dx = \sum_{n=1}^{\infty} \int_0^{\infty} xe^{-xn}\, dx$$

The function can be integrated now by applying the following substitutions,

$$u = xn$$
$$x = \frac{u}{n}$$
$$du = n\, dx$$
$$dx = \frac{1}{n}\, du$$

Now the integral becomes

$$\int_0^{\infty} \sum_{n=1}^{\infty} xe^{-xn}\, dx = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{ue^{-u}}{n^2}\, du$$

Factor out $\frac{1}{n^2}$ out of the integral,

$$\int_0^{\infty} \sum_{n=1}^{\infty} xe^{-xn}\, dx = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{\infty} ue^{-u}\, du$$

One can recognize the gamma function $\Gamma(s)$ represented as

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x}\, dx$$

for all numbers with a real part greater than 0. If z is set equal to 2, then $\Gamma(2)$ is shown by

$$\Gamma(2) = \int_0^{\infty} xe^{-x}\, dx$$

Returning to the previous integral, it can be written now as

$$\int_0^{\infty} \sum_{n=1}^{\infty} xe^{-xn}\, dx = \sum_{n=1}^{\infty} \frac{1}{n^2} \Gamma(2)$$

And considering that $\Gamma(z) = (z-1)!$, then $\Gamma(2) = 1! = 1$. Then,

$$\int_0^\infty \sum_{n=1}^{\infty} xe^{-xn}\, dx = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

The infinite series and product expansions of the sine function may be brought up,

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

Expanding them, one gets

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\sin z = z \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{(2\pi)^2}\right)\left(1 - \frac{z^2}{(3\pi)^2}\right)(\cdots)$$

$$\sin z = \left(z - \frac{1}{\pi^2} z^3\right)\left(1 - \frac{z^2}{(2\pi)^2}\right)\left(1 - \frac{z^2}{(3\pi)^2}\right)(\cdots)$$

$$\sin z = \left(z + \left(-\frac{1}{\pi^2} - \frac{1}{(2\pi)^2}\right) z^3 + \frac{1}{(2\pi^2)^2} z^5\right)\left(1 - \frac{z^2}{(3\pi)^2}\right)(\cdots)$$

$$\sin z = \left(z + \left(-\frac{1}{\pi^2} - \frac{1}{(2\pi)^2} - \frac{1}{(3\pi)^2} - \cdots\right) z^3 + (\cdots)\, z^5 + (\cdots)\, z^7 + \cdots\right)$$

From this one can set the following equal,

$$\left(-\frac{1}{\pi^2} - \frac{1}{(2\pi)^2} - \frac{1}{(3\pi)^2} - \cdots\right) z^3 = -\frac{z^3}{3!}$$

or simply

$$-\sum_{n=1}^{\infty} \frac{z^3}{(n\pi)^2} = -\frac{z^3}{3!}$$

Divide both sides by $-z^3$,

$$\sum_{n=1}^{\infty} \frac{1}{(n\pi)^2} = \frac{1}{3!}$$

$3! = 6$, and so

$$\sum_{n=1}^{\infty} \frac{1}{(n\pi)^2} = \frac{1}{6}$$

Multiply both sides by π^2,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Thus it has been proven that

$$\int_0^\infty \frac{x}{e^x - 1}\, dx = \frac{\pi^2}{6}$$

and therefore

$$\zeta(2) = \frac{\pi^2}{6}$$