Weekly Homework 3

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Math 4377: Algebraic Structures

September 12, 2014

Problem 1.
Find all Solutions to the equation $x^2 \oplus x = [0]$ in $\mathbb{Z}_4$

Proof. □

Problem 2.
(1) Prove: If $[a] \in \mathbb{Z}_n$ is a unit, then $[a]$ is not a zero divisor.
(2) Prove: If $[b] \in \mathbb{Z}_n$ is a zero divisor, then $[b]$ is not a unit.

Proof. □

Problem 3.
Show that every nonzero element of $\mathbb{Z}_n$ is either a unit or a zero divisor.

Proof. □

Problem 4.
Suppose that $[a]$ is a unit in $\mathbb{Z}_n$ and $[b]$ is an element of $\mathbb{Z}_n$. Prove that the equation $[a]x = b$ has exactly one solution in $\mathbb{Z}_n$

Proof. □

Problem 5.
Suppose that $[a]$ and $[b]$ are both units in $\mathbb{Z}_n$. Show that the product $[a] \cdot [b]$ is also a unit in $\mathbb{Z}_n$. (Note that this confirms closure under multiplication in the group U_n).

Proof. □

Problem 6.
Which of the following are Groups? Which of the following are not groups, and why?

(1) $G = \{2, 4, 6, 8\}$ in $\mathbb{Z}_{10}$. Where $a \star b = ab$
(2) $G = \mathbb{Q}^*$, where $a \star b = \frac{a}{b}$
(3) $G = \mathbb{Z}$, where $a \star b = a - b$
(4) $G = \{2^x \mid x \in \mathbb{Q}\}$, where $a \star b = ab$

Proof. □

Problem 7.
Consider the set $Q = \{\ \pm 1, \pm \text{i}, \pm \text{j}, \pm \text{k}\}$ of the complex matrices as follows:

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$i = \begin{bmatrix} i & 0 \\ 0 & \text{-}i \end{bmatrix}$$

$$j = \begin{bmatrix} 0 & 1 \\ \text{-}1 & 0 \end{bmatrix}$$

$$k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Show that Q is a group under matrix multiplication by writing out its multiplicaiton table. (Note: Q is called the quartenion group).

Proof. □