

CALCULUS

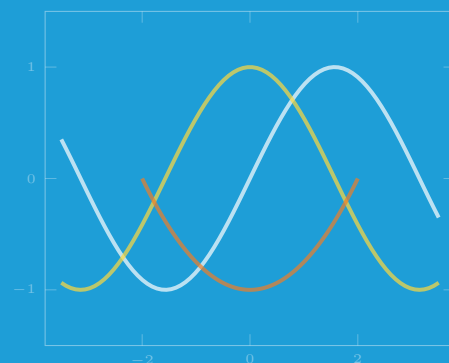
A modern course with
applications and visualizations

1 Limits and Continuity

2 Derivatives

3 Applications of Derivatives

4 Indefinite Integrals







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1

Limits and Continuity

In this chapter you will study the fundamental concept of a *limit*, which underpins all of differential and integral analysis. The main topics are:

- 1 Intuitive idea of a limit
- 2 Formal definition (ε - δ)
- 3 One-sided limits
- 4 Limits at infinity
- 5 Continuity

Did You Know?

Why are limits important? The derivative, the integral, and the sum of a series are all defined through a **limit** process. Without this concept, Calculus would not exist!

Lesson 1.1 — Intuitive Idea of a Limit

The **limit** of a function $f(x)$ as x approaches a describes the behavior of f *near* a , regardless of what happens *at* a .

- 1 Analyze the table of values below for $f(x) = \frac{x^2-1}{x-1}$:

x	$f(x)$	x	$f(x)$
0.9	1.9	1.1	2.1
0.99	1.99	1.01	2.01
0.999	1.999	1.001	2.001

- 2 Notice that the values approach **2** as $x \rightarrow 1$:

Limit Notation

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

1.1 Formal Definition of a Limit

* Definition — Limit (ε - δ Definition)

Let f be defined on a neighborhood of a (except possibly at a). We say that

$$\lim_{x \rightarrow a} f(x) = L$$

if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

► Example 1.1

Prove that $\lim_{x \rightarrow 3} (2x - 1) = 5$.

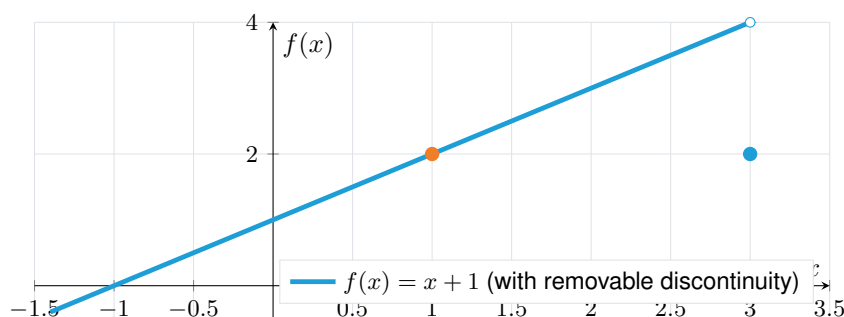
Solution:

Given $\varepsilon > 0$, take $\delta = \varepsilon/2$. If $0 < |x - 3| < \delta$, then

$$|(2x - 1) - 5| = 2|x - 3| < 2\delta = \varepsilon.$$

⚠ Caution!

The ε - δ definition requires the limit to exist on both sides of a . If the one-sided limits $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ differ, the limit **does not exist**.



Example of a function with a removable discontinuity at $x = 3$.

1.2 Properties of Limits

★ Theorem — Algebra of Limits

If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, then:

1. $\lim_{x \rightarrow a} [f(x) + g(x)] = L + M$

3. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$

2. $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = L \cdot M$

4. $\lim_{x \rightarrow a} [c f(x)] = cL$

★ Theorem — Squeeze Theorem (Sandwich Theorem)

If $g(x) \leq f(x) \leq h(x)$ on a neighborhood of a , and $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$, then $\lim_{x \rightarrow a} f(x) = L$.

► Example 1.2 — Fundamental Trigonometric Limit

Show that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

Idea: Use the Squeeze Theorem with $\cos x \leq \frac{\sin x}{x} \leq 1$ for $0 < |x| < \pi/2$.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

1.3 Continuity

✱ Definition — Continuity at a Point

A function f is **continuous at** a if:

1. $f(a)$ is defined;
2. $\lim_{x \rightarrow a} f(x)$ exists;
3. $\lim_{x \rightarrow a} f(x) = f(a)$.

✓ Continuous Function

Graph with no “jumps”
Can compute $f(a)$ directly
Theorems like IVT apply

✗ Discontinuous Function

Graph with “holes”, jumps or asymptotes
Limit may not exist
Requires careful analysis

Chapter 1 Exercises

1.1 — Direct computation

Evaluate the limits:

a) $\lim_{x \rightarrow 2} (3x^2 - x + 1)$

b) $\lim_{x \rightarrow 0} \frac{\tan x}{x}$

c) $\lim_{x \rightarrow \infty} \frac{2x^2 - 1}{x^2 + 3}$

d) $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$

1.2 — ε - δ Definition

Using the formal definition, prove that $\lim_{x \rightarrow 1} (4x - 3) = 1$.

1.3 — Continuity

Determine the values of k for which

$$f(x) = \begin{cases} kx^2 + 2, & x \leq 1 \\ 3x + k, & x > 1 \end{cases}$$

is continuous at $x = 1$.

Quick Answer Key

1a) 11
1b) 1

1c) 2
1d) $\frac{1}{4}$

2) $\delta = \varepsilon/4$
3) $k = \frac{1}{2}$

2

Derivatives

The **derivative** measures the instantaneous rate of change of a function. Geometrically, it is the slope of the tangent line to the graph.

● Did You Know?

The derivative appears in Physics as **velocity** ($v = ds/dt$), in Economics as **marginal cost**, and in Biology as a population growth rate.

- 1 **Average vs. instantaneous rate of change**
- 2 **Definition via limits**
- 3 **Differentiation rules**
- 4 **Chain rule**
- 5 **Higher-order derivatives**

Lesson 2.1 — From Average Rate to Derivative

- 1 The **average rate of change** of f on $[a, a + h]$ is:

$$\frac{\Delta f}{\Delta x} = \frac{f(a + h) - f(a)}{h}$$

- 2 Letting $h \rightarrow 0$, the average rate becomes the **derivative** at a :

Definition of the Derivative

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

- 3 The result is the **slope** of the tangent line at $(a, f(a))$.

2.1 Differentiation Rules

★ Theorem — Elementary Rules

1. $\frac{d}{dx}[c] = 0$
2. $\frac{d}{dx}[x^n] = nx^{n-1}$
3. $\frac{d}{dx}[cf] = cf'$
4. $\frac{d}{dx}[f + g] = f' + g'$
5. $\frac{d}{dx}[fg] = f'g + fg'$
6. $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{f'g - fg'}{g^2}$

★ Theorem — Chain Rule

If $y = f(u)$ and $u = g(x)$, then

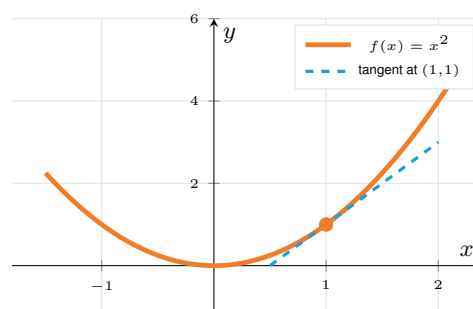
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \iff [f(g(x))]' = f'(g(x)) \cdot g'(x).$$

► Example 2.1

Compute $\frac{d}{dx}[\sin(x^2)]$.

$$u = x^2, \frac{du}{dx} = 2x, \frac{d}{du}[\sin u] = \cos u$$

$$2x \cos(x^2)$$



Function	Derivative	Note
$\sin x$	$\cos x$	trigonometric functions
$\cos x$	$-\sin x$	
e^x	e^x	unique function $f = f'$
$\ln x$	$1/x$	
x^n	nx^{n-1}	power rule
$\arctan x$	$\frac{1}{1+x^2}$	inverse trig.

Chapter 2 Exercises

2.1 — Basic rules

Differentiate each function:

a) $f(x) = 3x^4 - 5x^2 + 7$

b) $g(x) = \sqrt{x} \cdot e^x$

c) $h(x) = \frac{\ln x}{x^2}$

d) $p(x) = \sin^3(2x)$

2.2 — Tangent lineFind the equation of the tangent line to $f(x) = x^3 - 2x + 1$ at $x = 2$.**Quick Answer Key**

1a) $12x^3 - 10x$

1b) $\frac{e^x}{2\sqrt{x}} + \sqrt{x}e^x$

1c) $\frac{1-2\ln x}{x^3}$

1d) $6\sin^2(2x)\cos(2x)$

2) $y = 10x - 11$



3

Applications of Derivatives

In this chapter we use derivatives to **analyze** the behavior of functions: finding maxima, minima, inflection points and sketching graphs with precision.

- **1 Increasing and decreasing intervals**
- **2 Local extrema — Derivative tests**
- **3 Concavity and inflection**
- **4 Optimization**
- **5 L'Hôpital's Rule**

Caution!

The maximum and minimum tests require care! A critical point $f'(c) = 0$ can be a **maximum**, **minimum**, or **saddle point**. Use the second derivative to confirm.

3.1 Local Extrema

★ Theorem — First Derivative Test

Let c be a critical point of f ($f'(c) = 0$):

- If f' changes from $+$ to $-$ at $c \Rightarrow$ **local maximum**.
- If f' changes from $-$ to $+$ at $c \Rightarrow$ **local minimum**.
- If f' does not change sign $\Rightarrow$ **saddle point**.

★ Theorem — Second Derivative Test

$$f''(c) > 0 \Rightarrow \text{local minimum}; \quad f''(c) < 0 \Rightarrow \text{local maximum}.$$

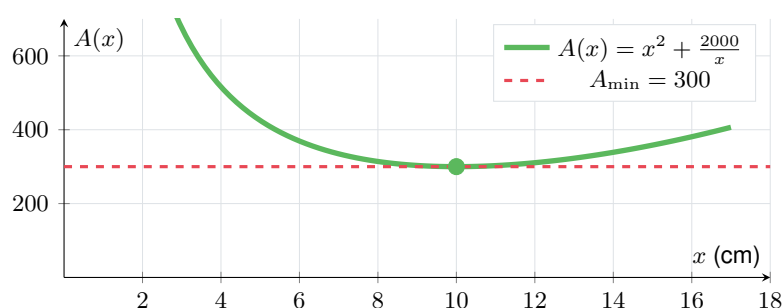
► Example 3.1 — Optimization

An open-top box with a square base of side x and height h must have volume $V = 500 \text{ cm}^3$. Minimize the surface area.

Solution: $A = x^2 + 4xh$, $h = 500/x^2$

$$A(x) = x^2 + \frac{2000}{x}, \quad A'(x) = 2x - \frac{2000}{x^2} = 0 \implies x = \sqrt[3]{1000} = 10$$

$$x = 10 \text{ cm}, h = 5 \text{ cm}, A_{\min} = 300 \text{ cm}^2$$



3.2 L'Hôpital's Rule

★ Theorem — L'Hôpital's Rule

If $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ (or $\pm\infty$) and $g'(x) \neq 0$ on a neighborhood of a , then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

► Example 3.2

$\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$ is of the form $0/0$. Applying L'Hôpital twice:

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{2x} = \lim_{x \rightarrow 0} \frac{e^x}{2} = \frac{1}{2}.$$

$$1/2$$

Chapter 3 Exercises

3.1 — Extrema

Find all local and global extrema of $f(x) = 2x^3 - 9x^2 + 12x - 4$ on $[0, 3]$.

3.2 — Optimization

A farmer wants to enclose a rectangular area of 3600 m^2 using the minimum length of fencing. What should the dimensions be?

3.3 — L'Hôpital

Evaluate: (a) $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$; (b) $\lim_{x \rightarrow \infty} xe^{-x}$.

Quick Answer Key

1) local min. $x = 2$; local max. $x = 1$; $f(1) = 1$
 global min. $f(0) = -4$; global max. **2) $60 \text{ m} \times 60 \text{ m}$ (square)**

3a) 3 3b) 0



4

Indefinite Integrals

The **indefinite integral** (or antiderivative) answers the question: “which function, when differentiated, gives $f(x)$?”

- **1 Antiderivatives and $\int$ notation**
- **2 Basic integrals**
- **3 Substitution (u -substitution)**
- **4 Integration by parts**
- **5 Partial fractions**

● Did You Know?

The **Fundamental Theorem of Calculus** links the integral to the derivative:

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

This result is considered one of the deepest in Mathematics!

Lesson 4.1 — Basic Integrals

Function $f(x)$	$\int f(x) dx$	Condition
x^n	$\frac{x^{n+1}}{n+1} + C$	$n \neq -1$
$\frac{1}{x}$	$\ln x + C$	$x \neq 0$
e^x	$e^x + C$	
$\sin x$	$-\cos x + C$	
$\cos x$	$\sin x + C$	
$\sec^2 x$	$\tan x + C$	
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$	$ x < 1$
$\frac{1}{1+x^2}$	$\arctan x + C$	

4.1 Substitution (u -substitution)

★ Theorem — Substitution Technique

If $u = g(x)$, then $du = g'(x) dx$ and

$$\int f(g(x)) g'(x) dx = \int f(u) du.$$

► Example 4.1

Evaluate $\int 2x \cos(x^2) dx$.

Solution: $u = x^2$, $du = 2x dx$

$$\int \cos u du = \sin u + C = \sin(x^2) + C.$$

► Example 4.2

Evaluate $\int \frac{x}{x^2 + 1} dx$.

Solution: $u = x^2 + 1$, $du = 2x dx$

$$\frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{\ln(x^2 + 1)}{2} + C.$$

4.2 Integration by Parts

★ Theorem — Integration by Parts

Formula (product rule for integration)

$$\int u dv = uv - \int v du$$

LIATE Rule for choosing u : **L**ogarithmic · **I**nverse trig. · **A**lgebraic · **T**rigonometric · **E**xponential.

► Example 4.3

Evaluate $\int x e^x dx$.

$u = x$, $dv = e^x dx \Rightarrow du = dx$, $v = e^x$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C = e^x(x - 1) + C.$$

$$e^x(x - 1) + C$$

Chapter 4 Exercises

4.1 — Basic integrals

Evaluate:

a) $\int (3x^2 - 2x + 5) dx$

b) $\int \frac{dx}{\sqrt{4 - x^2}}$

c) $\int e^{3x} dx$

d) $\int \sec^2(5x) dx$

4.2 — SubstitutionEvaluate $\int \frac{\ln x}{x} dx$.**4.3 — By parts**Evaluate $\int x^2 \ln x dx$.**Quick Answer Key**

1a) $x^3 - x^2 + 5x + C$

1b) $\arcsin(x/2) + C$

1c) $\frac{1}{3}e^{3x} + C$

1d) $\frac{1}{5}\tan(5x) + C$

2) $\frac{(\ln x)^2}{2} + C$

3) $\frac{x^3}{3}\ln x - \frac{x^3}{9} + C$



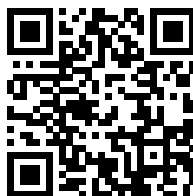
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**“Calculus is the language in which
God wrote the Universe.”**

— Galileo Galilei (attributed)



Wolfram Alpha —
explore functions online

